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Inhomogeneous R^2 Inflationary Models

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We discuss two inhomogeneous R^2 inflationary models, a spherically symmetric model and a Szekers class II model. We analyze the behavior of inflation in these systems and find the exact solutions. In the large-time limit, the metric can be written for the de Sitter solution.

1. INTRODUCTION

Is our observed universe homogeneous and isotropic on very large scales? In this paper we examine inflation in inhomogeneous universes. In the standard hot big bang model homogeneity and isotropy are put in by hand as initial conditions, but with no explanation. The inflationary universe (Guth, 1981; Linde, 1982; Albrecht and Steinhart, 1982; Abbott and Wise, 1984; Lucchin and Matarrese, 1985; Mathiazhagan and Johri, 1984; La and Steinhardt, 1989; Mijic *et al.*, 1986; Berkin, 1990; Barrow, 1990; Barrow and Ottewill, 1983; Maeda, 1988) is regarded as one of the best explanations, and has been investigated widely in the past decade. So far there are two basic types of inflationary models. One type introduces a scalar field into the Lagrangian action (Guth, 1981; Linde, 1982; Albrecht and Steinhart, 1982; Abbott and Wise, 1984; Lucchin and Matarrese, 1985), Mathiazhagan and Johri, 1984; La and Steinhardt, 1989), namely that of gravity coupled minimally to a real scalar field. Old inflation (Guth, 1981) new inflation (Linde, 1982; Albrecht and Steinhart, 1982), chaotic inflation (Abbott and Wise, 1984; Lucchin and Matarrese, 1985), and extended inflation (Mathiazhagan and Johri, 1984; La and Steinhardt, 1989) take place with difference scalar potential and initial values.

Another type is based on higher-derivative gravity (Mijic *et al.*, 1986; Berkin, 1990; Barrow and Ottewill, 1983; Barrow, 1987; Maeda, 1988) or

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a cosmological constant (Wald, 1983; Jensen and Stein-Schabes, 1987). In this type an interesting mechanism dubbed R^2 inflation adds the R^2 term to the Lagrangian action; it has been studied by many authors. However, these models are described within the framework of the homogeneous Bianchi types, even the homogeneous and isotropic FRW cosmology. Hence the homogeneity is still put in by hand. The reasons for this are purely technical.

Recently, several authors have studied inflation in homogeneous space-time. Stein-Schabes (1987) has presented the exact inflationary solutions in a spherically symmetric, inhomogeneous metric in the presence of a massless scalar field with flat potential, and discussed the process of isotropization and homogenization in detail. Carone and Guth (1990) have analyzed inhomogeneous models with potential

$$V(\phi)=A[\phi^4 \ln(\phi^2/v^2)+(v^4-\phi^4)/2]$$

Jensen and Stein-Schabes (1987) have proved that inflation exists in inhomogeneous space-time with a positive cosmological constant.

But R^2 inflation in inhomogeneous space-time has not been studied in detail. In this paper we discuss the R^2 inflation in inhomogeneous space-time with spherical symmetry and in inhomogeneous Szekers class II space-time.

2. INHOMOGENEOUS CASE WITH SPHERICAL SYMMETRY

We start with the Lagrangian density $L=R+\varepsilon R^2+L_m$, where R is the Ricci scale and $\varepsilon\neq 0$. The field equations are

$$(1+2\varepsilon R)(R_{ab}-g_{ab}R/2)=-2\varepsilon g_{ab}(R^2/4+\Box R)+2\varepsilon R_{;ab}+T_{ab} \quad (2.1)$$

where units are taken with $8\pi G=C=1$, $T_{ab}=(\rho+p)u_a u_b+pg_{ab}$.

Now we consider the Tolman–Bondi line element (Stein-Schabes, 1987)

$$ds^2=-dt^2+X^2(r,t)\,dr^2+Y^2(r,t)(d\theta^2+\sin^2\theta\,d\varphi^2) \quad (2.2)$$

In this metric, the field equations can be written as

$$(1+2\varepsilon R)[2\dot{X}\dot{Y}/XY+\dot{Y}^2/Y^2+1/Y^2-(2Y''/Y+Y'^2/Y^2-2X'Y'/XY)/X^2]$$
$$=2\varepsilon(R^2/4+\Box R)+2\varepsilon R_{;tt}+\rho \quad (2.3)$$

$$(1+2\varepsilon R)(2\ddot{Y}/Y+\dot{Y}^2/Y^2+1/Y^2-Y'^2/X^2Y^2)$$
$$=2\varepsilon(R^2/4+\Box R)-2\varepsilon R_{;rr}-p \quad (2.4)$$

$$(1+2\varepsilon R)[\ddot{X}/X+\ddot{Y}/Y+\dot{X}\dot{Y}/XY-(Y''/Y-X'Y'/XY)/X^2]$$
$$=2\varepsilon(R^2/4+\Box R)-2\varepsilon R_{;\theta\theta}-p \quad (2.5)$$

$$2(1+2\varepsilon R)(\dot{X}Y'/XY-\dot{Y}'/Y)=2\varepsilon R_{;tr} \quad (2.6)$$

As is usually the case, the system of partial differential equations is overdetermined. So some of the equations will be regarded as dynamical and the rest as constraint equations.

We assume

$$X(r, t) = X_r(r)X_t(t), \qquad Y(r, t) = Y_r(r)Y_t(t) \tag{2.7}$$

and $Y_r' \neq 0$. Equation (2.6) can be written as

$$\begin{aligned} &2[1+2\varepsilon h(t)](\dot{X}_t/X_t - \dot{Y}_t/Y_t) \\ &\quad = 2\varepsilon[12\dot{Y}_t/Y_t^3Y_r^2 + 4(\dot{X}_t/X_t - \dot{Y}_t/Y_t)f/X_t^2 + 6X_tY_rf'/X_t^3Y_r'] \end{aligned} \tag{2.8}$$

where

$$h(t) = 2(\ddot{X}_t/X_t + 2\ddot{Y}_t/Y_t + 2\dot{X}_t\dot{Y}_t/X_tY_t) \tag{2.9}$$

$$f(r) = (2Y_r''/Y_r + Y_r'^2/Y_r - 2X_r'Y_r')/X_r^2 \tag{2.10}$$

The overdot signifies $\partial/\partial t$ and the prime $\partial/\partial r$.

Because the left of equation (2.8) is only dependent on time, but the right is dependent on time and space, we have only the following solutions:

$$\dot{X}_t/X_t = \dot{Y}_t/Y_t \tag{2.11}$$

$$1/Y_r^2 = f + k \tag{2.12}$$

where k is an arbitrary integer.

From equation (2.12) we can get the space solution of the metric,

$$X_r = Y_r'/(1 - kY^2/3)^{1/2} \tag{2.13}$$

Considering equations (2.3) and (2.5), the time part of the solution is dependent on

$$\ddot{R} + 3H\dot{R} + R/6\varepsilon = (4 - 3\gamma)\rho/6\varepsilon \tag{2.14}$$

$$\dot{R} = R^2/12H - HR - H/2\varepsilon - k/6\varepsilon HX_t^2 - kR/3HX_t^2 + \rho/6\varepsilon H \tag{2.15}$$

where

$$R = 6\dot{H} + 12H^2 + 2k/X_t^2 \tag{2.16}$$

and H is the Hubble parameter,

$$H = \dot{X}_t/X_t \tag{2.17}$$

Equations (2.14) and (2.15) are the same as those in a homogeneous Robertson–Walker metric ($k/3 = -1, 0, +1$, corresponding to open, flat, closed models).

1. We first consider the case with no matter. Using the method of Berkin (1990), we obtain

$$\dot{H}=0 \Rightarrow R=12H^2+2k/X_t^2 \tag{2.18}$$

$$\dot{R}=0 \Rightarrow R=(6H^2+2k/X_t^2)\{1+[1+1/(6H^2+2k/X_t^2)]^{1/2}\} \tag{2.19}$$

When $k=0$ (flat), Berkin has shown that for $\varepsilon \to \infty$ or ε finite but large, only if any universe satisfies some initial conditions does exponential inflation take place. Mijic *et al.* (1986) have studied this case in detail. If $k \neq 0$ (open or closed), these two relations are not identical even for $\varepsilon \to \infty$. But we know that once the universe starts to inflate, $2k/X_t^2$ will rapidly disappear, and these equations approach to the case of $k=0$. So in the case of $k \neq 0$, if $\varepsilon \to \infty$ or ε finite but large, exponential inflation exists also.

2. Now we consider the presence of matter. The energy-momentum conservation implies

$$\rho \sim X_t^{-3\gamma} \tag{2.20}$$

When $\gamma > 0$, the influence of matter will be lost rapidly after inflation. So the above results remain the same.

When $\gamma = 0$, from equations (2.14) and (2.15) we find

$$R=-1/2\varepsilon; \qquad \varepsilon=-1/8\rho \tag{2.21}$$

The time part of the solution is given by

$$X=\begin{cases}(k/\rho)^{1/2}\sinh[(\rho/3)^{1/2}t+X_0] & k>0\\ X_0\exp[(\rho/3)^{1/2}t] & k=0\\ (-k/\rho)^{1/2}\cosh[(\rho/3)^{1/2}t+X_0] & k<0\end{cases} \tag{2.22}$$

where X_0 is an integer.

Clearly $X_t \sim \exp[(\rho/3)^{1/2}t]$ for large time. That is, in the large-time limit, the time part of the metric becomes the de Sitter form.

Finally, we can construct the full solutions,

$$ds^2=-dt^2+X_t^2[X_r^2\,dr^2+Y_r^2(d\theta^2+\sin^2\theta\,d\varphi^2)] \tag{2.23}$$

Introducing a new radial variable $Z=Y_r$ and a new constant $\eta=k/3$, we then get

$$ds^2=-dt^2+X_t^2[dZ^2/(1-\eta Z^2)+Z^2(d\theta^2+\sin^2\theta\,d\varphi^2)] \tag{2.24}$$

This is the exact Robertson–Walker metric in its open ($\eta=1$), flat ($\eta=0$), or closed ($\eta=1$) versions. In the large-time limit, all cases become the exact de Sitter solution. The results are similar to those in the presence of a homogeneous scalar field in spherically symmetric inhomogeneous space-time (Stein-Schabes, 1987).

So even if we assume that the metric is more general than the standard model, the equations only accept the isotropic and homogeneous solution.

3. INHOMOGENEOUS SZEKERS CLASS II SPACE-TIME

The Szekers class II metric can be written as

$$ds^2 = -dt^2 + Q^2(t, x, y, z)\, dx^2 + L^2(t)(dy^2 + dz^2) \tag{3.1}$$

According to Berman (1990),

$$Q(t, x, y, z) = L(t) \cdot Q_s(x, y, z) \tag{3.2}$$

The field equations can be written as

$$(1 + 2\varepsilon R)(3\dot{L}^2/L^2 - f/L^2) = 2\varepsilon(R^2/4 + \Box R) + 2\varepsilon R_{;00} + \rho \tag{3.3}$$

$$(1 + 2\varepsilon R)(-2\ddot{L}/L - \dot{L}^2/L) = -2\varepsilon(R^2/4 + \Box R) + 2\varepsilon R_{;11} + p \tag{3.4}$$

$$(1 + 2\varepsilon R)(-2\ddot{L}/L - \dot{L}^2/L^2 + Q_{s,33}/L^2 Q_s) = -2\varepsilon(R^2/4 + \Box R) + 2\varepsilon R_{;22} + p \tag{3.5}$$

$$(1 + 2\varepsilon R)(-2\ddot{L}/L - \dot{L}^2/L^2 + Q_{s,22}/L^2 Q_s) = -2\varepsilon(R^2/4 + \Box R) + 2\varepsilon R_{;33} + p \tag{3.6}$$

$$3\dot{L} f_{,1}/L^3 = 0 \tag{3.7}$$

$$f_{,2} = f_{,3} = 0 \tag{3.8}$$

$$f_{,12} = f_{,1} Q_{s,2}/Q_s; \qquad f_{,13} = f_{,1} Q_{s,3}/Q_s \tag{3.9}$$

$$(1 + 2\varepsilon R) Q_{s,23}/Q_s = -4\varepsilon f_{,23}/L^2 \tag{3.10}$$

where

$$R = 6(\ddot{L}/L + \dot{L}^2/L^2) - 2(Q_{s,22} + Q_{s,33})/L^2 Q_s$$

$$f(x, y, z) = (Q_{s,22} + Q_{s,33})/Q_s$$

From equations (3.7) and (3.8), we easily find $f = k$. So R is dependent only on time, and we get $R_{;22} = R_{;33}$. Using these results in equations (3.5) and (3.6), we have

$$Q_{s,22} = Q_{s,33}$$

On the other hand, taking $f=k$ in (3.10), we get two cases:

1. $R=-1/2\varepsilon$. In this case, from equations (3.3) and (3.4), we know that $\varepsilon=-1/8\rho$, $\gamma=0$; here γ is given from the state equation $p=(\gamma-1)\rho$. We can find the solutions

$$Q_s(x,y,z)=\begin{cases}\phi_1(x)\exp[(k/2)^{1/2}(y+z)] \\ \quad +\phi_2(x)\exp[-(k/2)^{1/2}(y+z)], & k>0 \\ yz\phi_1(x)+y\phi_2(x)+z\phi_3(x)+\phi_4(x), & k=0 \\ \phi_1(x)\sin[(-k/2)^{1/2}(y+z)] \\ \quad +\phi_2(x)\cos[(-k/2)^{1/2}(y+z)], & k<0\end{cases} \tag{3.11}$$

and

$$L(t)=\begin{cases}(k/\rho)^{1/2}\sinh[(\rho/3)^{1/2}t+L_0], & k>0 \\ L_0\exp[(\rho/3)^{1/2}t], & k=0 \\ (-k/\rho)^{1/2}\cosh[(\rho/3)^{1/2}t+L_0], & k<0\end{cases} \tag{3.12}$$

where $\phi_i(x)$ $(i=1,2,3,4)$ are arbitrary functions of x, and L_0 is the integration constant.

From the above solutions, we know that the time part of the solution is exactly the de Sitter solution when $k=0$. If $\phi_1(x)=0$, this solution is the same as in Berman (1990). For large time we get $L(t)\sim\exp[(\rho/3)^{1/2}t]$ when $k\neq 0$, i.e., the time part of the solution approaches the de Sitter form.

2. $Q_{s,23}=0$

In this case, we easily show that $k=0$ and get the space solution

$$Q_s(x,y,z)=y\phi_1(x)+z\phi_2(x)+\phi_3(x) \tag{3.13}$$

The equations of the time part are

$$\ddot{R}+3H\dot{R}+R/6\varepsilon=(4-3\gamma)\rho/6\varepsilon \tag{3.14}$$

$$\dot{R}=R^2/12H-RH-H/2\varepsilon+\rho/6\varepsilon H \tag{3.15}$$

and with

$$R=6(\ddot{L}/L+\dot{L}^2/L^2)=6\dot{H}+12H^2 \tag{3.16}$$

These equations are the same as in flat RW models (Linde, 1982; Albrecht and Steinhart, 1982; Mijic *et al.*, 1986). Berkin has shown that for $\varepsilon\to\infty$ or ε is finite but large, if a universe satisfies some initial conditions, exponential inflation exists. If matter is included, these results remain. In fact, the energy-momentum conservation implies

$$\rho \sim 1/L^{3\gamma} \tag{3.17}$$

Therefore if $\gamma > 0$, once the universe starts to inflate, the matter will rapidly lose any influence it may have had.

This work shows that R^2 inflation exists in inhomogeneous space-time.

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